SYMBΩL

MAB241 COMPLEX VARIABLES

CAUCHY'S INTEGRAL FORMULA

1 The formula

Theorem 2.3 *Let f be analytic everywhere on and within a positively oriented simple closed contour C. For any point z_0 interior to C*

$$f(z_0) = \frac{1}{2\pi \mathrm{i}} \int_C \frac{f(z)}{z - z_0} \, \mathrm{d}z. \tag{1}$$

The proof of this theorem is given in the lecture notes.
This formula may appear to be a very complicated way of calculating $f(z_0)$, but by rearranging this result we get

$$\int_C \frac{f(z)}{z - z_0} \, \mathrm{d}z = 2\pi \mathrm{i} f(z_0),$$

which is a way of evaluating an integral along a closed curve which is not analytic just by evaluating a function at a certain point. This handout will show you how to use this rearrangement of Cauchy's integral formula to evaluate certain integrals.

2 When to use Cauchy's integral formula

Questions where you will need to use Cauchy's integral formula will not always be in the form shown in equation (1), sometimes you will need to use algebraic manipulation to get an integral in the same form as the theorem.

2.1 Example

Let $C_1(1)$ be the positively oriented circle of radius 1 centred on point $z = 1$. Calculate

$$I = \int_{C_1(1)} \frac{z}{z^2 - 1} \, \mathrm{d}z. \tag{2}$$

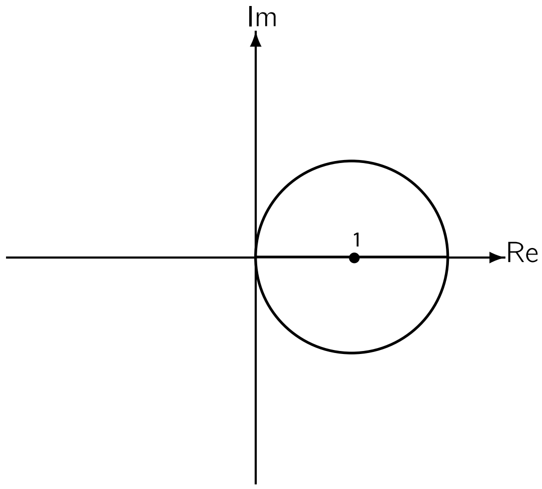
Solution

This integrand has singularities at $z = \pm 1$ and is not analytic inside the closed contour so we cannot use Cauchy's integral formula yet, and it would be very difficult to attempt solving this using parameterisation. However, we can use simple algebra to manipulate this into something we can recognise as suitable for Cauchy's integral formula.

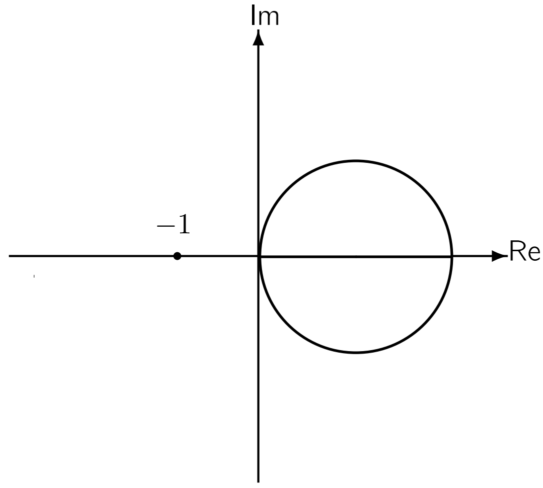
$$I = \int_{C_1(0)} \frac{z}{z^2 - 1} \, \mathrm{d}z = \int_{C_1(0)} \frac{z}{(z + 1)(z - 1)} \, \mathrm{d}z.$$

At this point we can split the integrand in two ways to find a possible $f(z)$, but a condition of Cauchy's integral formula is that f must be analytic everywhere on and within closed contour C. We can draw the contour on an Argand diagram to see if either of the functions have singularities within the closed contour.

singularities for $\frac{z}{z-1}$



singularities for $\frac{z}{z+1}$



From the diagrams we can see that $\frac{z}{z-1}$ is not analytic within the given contour as it has a singularity in it. Now we split the integrand into a function $f(z)$ multiplied by a $\frac{1}{z-z_0}$ term, to get

$$I = \int_{C_1(0)} \frac{z}{z+1}\frac{1}{z-1}\,\mathrm{d}z,$$

where we have taken

$$f(z) = \frac{z}{z+1}, \quad z_0 = 1.$$

$$\int_{C_1(1)} \frac{z}{z+1}\frac{1}{z-1}\,\mathrm{d}z = \int_C \frac{f(z)}{z-1}\,\mathrm{d}z = 2\pi\mathrm{i}f(1) = 2\pi\mathrm{i}\frac{1}{1+1} = \pi\mathrm{i}.$$

Using some simple algebra and Cauchy's integral formula, we have found

$$\int_{C_1(1)} \frac{z}{z^2-1}\,\mathrm{d}z = \pi\mathrm{i}.$$

3 Key points

- Questions where you will expected to use Cauchy's integral formula will not always be obvious, try to change the function so you can easily spot the $f(z)$ and z_0 terms.
- Remember the conditions for the theorem, make sure that your chosen z_0 is interior to C, and the function $f(z)$ is analytic everywhere inside C.
- Evaluating $f(z_0)$ and multiplying by $2\pi\mathrm{i}$ will give you the value of the integral.

For more information on Cauchy's integral Formula refer to section 2.5 in the lecture notes.